Morphing and barycenters

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wikipedia definition

Morphing is a special effect in motion pictures and animations that changes (or morphs) one image into another through a seamless transition. Most often it is used to depict one person turning into another through technological means or as part of a fantasy or surreal sequence. Traditionally such a depiction would be achieved through cross-fading techniques on film. Since the early 1990s, this has been replaced by computer software to create more realistic transitions.

Pictures'morphing

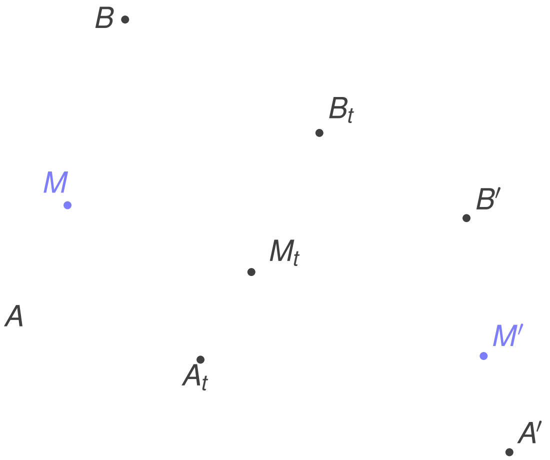
- http://youtube.com/watch?v=nUDIoN-_Hxs

Curves' morphings

From now on we shall leave pictures' morphing and deal exclusively with curves's morphing :

- segments's morphing
- triangles's morphing
- parallelograms's morphing
- squares's morphing
- nice curves's morphing

Principle of morphing



The curve AB is the source, the curve A'B' is the target.
t is the morphing parameter.
When M_θ runs along the curve AB and M' along the curve A'B', M_t runs along the intermediate curve A_tB_t.
When t varies from 0 to 1, the curve A_tB_t goes from the curve AB to the curve A'B'.
In the following cases, θ is a barycental parameter.

Two points' barycenter

- We say that G is a barycenter of two given points A and B with respective given masses a and b if

$$a\overrightarrow{GA} + b\overrightarrow{GB} = \vec{0}$$

$$
\begin{aligned}
a\overrightarrow{GA} + b\overrightarrow{GB} = \vec{0} \iff & (a+b)\overrightarrow{AG} = b\overrightarrow{AB} \\
\iff & a\overrightarrow{OA} + b\overrightarrow{OB} = (a+b)\overrightarrow{OG}
\end{aligned}
$$

whatever be the point O.

If $a + b \neq 0$,

$$a\overrightarrow{GA} + b\overrightarrow{GB} = \vec{0} \iff \overrightarrow{AG} = \frac{b}{a+b}\overrightarrow{AB}$$
$$\implies G \text{ is unique}$$

Barycenter of three or more points

Let A, B and C be three points
with respective masses a, b and c.
G is a barycenter of $(A, a), (B, b), (C, c)$ if

$$a\,\overrightarrow{GA} + b\,\overrightarrow{GB} + c\,\overrightarrow{GC} = \vec{0}$$

Barycenter of three or more points

Whatever be the point O

$$a\,\overrightarrow{GA} + b\,\overrightarrow{GB} + c\,\overrightarrow{GC} = \vec{0}$$
$$\iff \quad (a+b+c)\,\overrightarrow{OG} = a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC}$$

If $a + b + c \neq 0$, G is unique.

And so on …

Elementary barycenter's properties

- The following diagram

$$(A, a) \qquad\qquad\qquad (B, b)$$

$$\searrow \quad \swarrow$$

$$(G, a + b)$$

- has to be inderstood as
- G is a barycenter of $(A, a), (B, b)$

Elementary barycenter's properties

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Exchangeability

$$
\begin{array}{ccc}
(A, a) & & (B, b) \\
& \searrow \quad \swarrow & \\
& (G, a+b) &
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{ccc}
(B, b) & & (A, a) \\
& \searrow \quad \swarrow & \\
& (G, a+b) &
\end{array}
$$

Exchangeability

$$\begin{array}{ccccc} (A,a) & & & & (B,b) \\ & \searrow & & \swarrow & \\ & & (G,a+b) & & \end{array} \iff \begin{array}{ccccc} (G,a+b) & & & & (B,-b) \\ & \searrow & & \swarrow & \\ & & (A,a) & & \end{array}$$

Homogeneity

For every real number $\lambda \neq 0$,

$$\begin{array}{ccc} (A, a) & & (B, b) \\ & \searrow \quad \swarrow & \\ & (G, a+b) & \end{array} \iff \begin{array}{ccc} (A, \lambda a) & & (B, \lambda b) \\ & \searrow \quad \swarrow & \\ & (G, \lambda(a+b)) & \end{array}$$

Barycenter's associativity : example

- G is a barycenter of $(A, a), (B, b), (C, c), (D, d)$
  if and only if
  it is a barycenter of $(H, a + b), (K, c + d)$
- where H is a barycenter of $(A, a), (B, b)$
  and K is a barycenter of $(C, c), (D, d)$.
- (A, a) (B, b) (C, c) (D, d)

  $(H, a + b)$ $(K, c + d)$

  (G, s)

  where $s = a + b + c + d$

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- $$\begin{array}{ccccccc} (A,a) & & (B,b) & & (C,c) & & (D,d) \\ & \searrow \; \swarrow & & & & \searrow \; \swarrow & \\ & (H, a+b) & & & & (K, c+d) & \\ & & \searrow & & \swarrow & & \\ & & & (G, s) & & & \end{array}$$

  where $s = a + b + c + d$

Segment

Let

$$
\begin{array}{ccc}
(A, 1-\alpha) & & (B, \alpha) \\
& \searrow \quad \swarrow & \\
& (M, 1) &
\end{array}
$$

- When α increases from 0 to 1, M runs along $[AB]$, from A to B.
- $\overrightarrow{AM} = \alpha \overrightarrow{AB}$

Segment

Let

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Segment

Let

- $$(A, 1-\alpha) \quad (B, \alpha) \\ \searrow \swarrow \\ (M, 1)$$

- When α increases from 0 to 1, M runs along $[AB]$, from A to B.
- $\overrightarrow{AM} = \alpha \overrightarrow{AB}$

- Let $[AB]$ and $[A', B']$ two segments.
- Assume $0 \leqslant \theta \leqslant 1$

$$(A, 1-\theta) \quad (B, \theta) \longrightarrow (M, 1) \qquad (A', 1-\theta) \quad (B', \theta) \longrightarrow (M', 1)$$

- M is the generic point of $[AB]$, M' the generic point of $[A'B']$
- Both for $[AB]$ and $[A'B']$, θ is the usual parameter related to barycenters ; this is a choice among many other choices of parameters.

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$$(A, 1-\theta) \quad (B, \theta) \qquad (A', 1-\theta) \quad (B', \theta)$$
$$\searrow \swarrow \qquad\qquad \searrow \swarrow$$
$$(M, 1) \qquad\qquad (M', 1)$$

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Segments's morphing : $[AB] \longrightarrow [A'B']$

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$$
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& \searrow \quad \swarrow & \\
& (M, 1) &
\end{array}
\qquad
\begin{array}{ccc}
(A', 1-\theta) & & (B', \theta) \\
& \searrow \quad \swarrow & \\
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$$\begin{array}{ccccccc} (A, 1-\theta) & & (B, \theta) & \quad (A', 1-\theta) & & (B', \theta) \\ & \searrow \quad \swarrow & & & \searrow \quad \swarrow & \\ & (M, 1) & & & (M', 1) & \end{array}$$

- M is the generic point of $[AB]$, M' the generic point of $[A'B']$
- Both for $[AB]$ and $[A'B']$, θ is the usual parameter related to barycenters ; this is a choice among many other choices of parameters.

- Morphing : let t, $0 \leqslant t \leqslant 1$ be the morphing parameter.

$(A, 1-t)$ ⟍ ⟋ (A', t) → $(A_t, 1)$ $\qquad$ $(B, 1-t)$ ⟍ ⟋ (B', t) → $(B_t, 1))$

$(M, 1-t)$ ⟍ ⟋ (M', t) → $(M_t, 1)$

Segments's morphing : $[AB] \longrightarrow [A'B']$

- The locus of M_t for fixed t when θ increases from 0 to 1 is the moving image of $[AB]$ during the morphing when t varies.
- For $t = 0$, it is $[AB]$; for $t = 1$, it is $[A'B']$.
- For every $t \in]0,\ 1[$, it is a segment.

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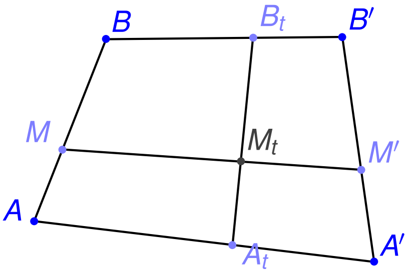
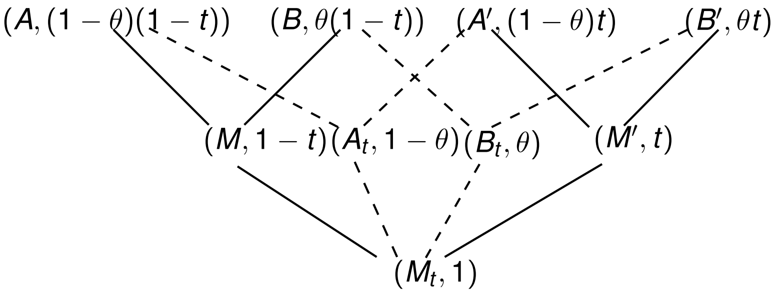
Illustration



Problem

We now prove that the running curve is a segment

Proof


$(A, (1-\theta)(1-t))$
$(B, \theta(1-t))$
$(A', (1-\theta)t)$
$(B', \theta t)$
$(M, 1-t)$
$(A_t, 1-\theta)$
(B_t, θ)
(M', t)
$(M_t, 1)$

Consequences

- $\Longrightarrow \overrightarrow{A_t M_t} = \theta \overrightarrow{A_t B_t}$
- $M_t \in [A_t B_t]$

Consequences

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- $M_t \in [A_t B_t]$

Parallelograms's morphing

A morphing from a parallelogram $ABCD$ to a parallelogram $A'B'C'D'$ produces a moving parallelogram.

- Let $ABCD$ and $A'B'C'D'$ be parallelograms
- Let us define C_t and D_t like A_t and B_t previously

$$(C, 1-t) \quad (C', t) \quad\quad (D, 1-t) \quad (D', t)$$

$$(C_t, 1) \quad\quad (D_t, 1)$$

- then, $A_tB_tC_tD_t$ is still a parallelogram.

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- Let $ABCD$ and $A'B'C'D'$ be parallelograms
- Let us define C_t and D_t like A_t and B_t previously

$$(C, 1-t) \searrow (C_t, 1) \swarrow (C', t) \qquad (D, 1-t) \searrow (D_t, 1) \swarrow (D', t)$$

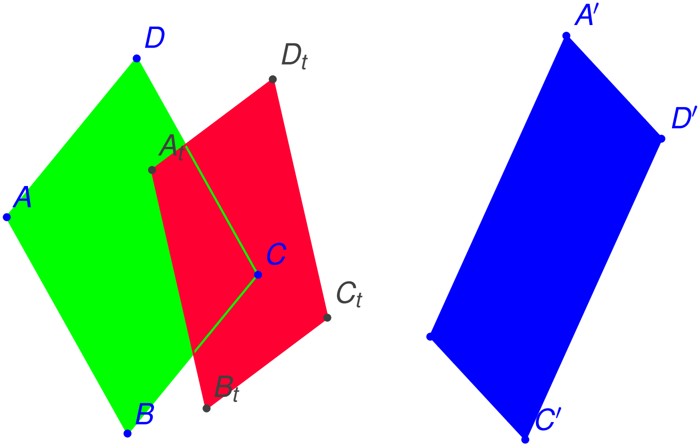
- then, $A_tB_tC_tD_t$ is still a parallelogram.

Parallelograms's morphing

A morphing from a parallelogram $ABCD$ to a parallelogram $A'B'C'D'$ produces a moving parallelogram.

- Let $ABCD$ and $A'B'C'D'$ be parallelograms
- Let us define C_t and D_t like A_t and B_t previously

  $(C, 1-t)$ ↘ ↙ (C', t) → $(C_t, 1)$ $(D, 1-t)$ ↘ ↙ (D', t) → $(D_t, 1)$
- then, $A_tB_tC_tD_t$ is still a parallelogram.


D
D_t
A_t
A
C
C_t
B_t
B
A'
D'
C'

Proof

- one remark :

$$ABCD \text{ parallelogram}$$

is equivalent to

$$\overrightarrow{AB} = \overrightarrow{DC}$$

is equivalent to

$$\overrightarrow{AB} = \overrightarrow{DA} + \overrightarrow{AC}$$

is equivalent to

$$\overrightarrow{AB} - \overrightarrow{AC} + \overrightarrow{AD} = \vec{0}$$

is equivalent to

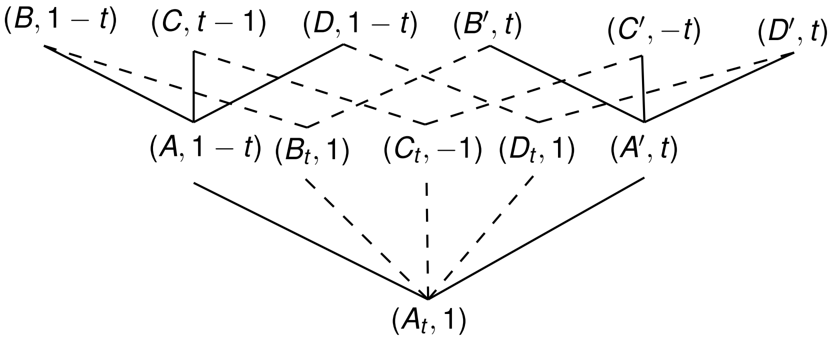
$$A = \mathrm{bar}\{(B;1),(C;-1),(D;1)\}$$

By homogeneity

$$
\begin{array}{ccc}
(B,1) & (C,-1) & (D,1) \\
 & \diagdown \, | \, \diagup & \\
 & (A,1) &
\end{array}
\qquad
\begin{array}{ccc}
(B',1) & (C',-1) & (D',1) \\
 & \diagdown \, | \, \diagup & \\
 & (A',1) &
\end{array}
$$

$$
\begin{array}{ccc}
(B,1-t) & (C,t-1) & (D,1-t) \\
 & \diagdown \, | \, \diagup & \\
 & (A,1-t) &
\end{array}
\qquad
\begin{array}{ccc}
(B',t) & (C',-t) & (D',t) \\
 & \diagdown \, | \, \diagup & \\
 & (A',t) &
\end{array}
$$

By associativity



- five points -> five points morphing
- segment -> segment morphing (with non barycentral parameters)
- segment -> segment morphing
- quadrilateral -> quadrilateral morphing
- square -> circle morphing
- heart -> diamond morphing

scilab

- square-circle's morphing
- hearts-circle's morphing
- four petals's flowers-circle's morphing
- apple-four petals's flower's morphing

historical morphing

- Louis-Philippe’s caricature